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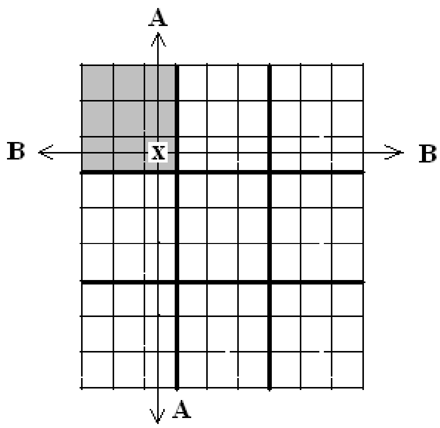
Sudoku, math and scheduling

There is more to the popular SUDOKU puzzle than amusement, says S.Ananthanarayanan.

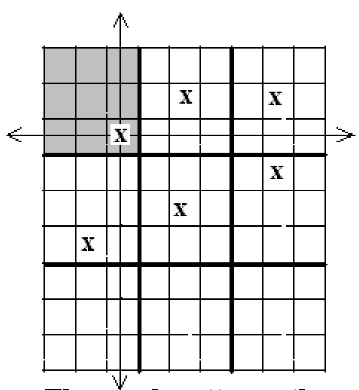
The puzzle represents the complex optimization problem, of making do with the least resources, that nature, technologists, managers, school principals face every day.

Sudoku

Sudoku is a puzzle of placing the numbers from 1 to 9 in a framework of 9x9 squares so that they satisfy a rule – first that a number occurs only once in the row and the column that it lies in and second, that it occurs only once in the 9 smaller squares drawn inside the main grid – as explained in the picture. As the player puts numbers in place, thus blocking rows, column and sub-squares, finding suitable numbers for left over places gets progressively more difficult and Sudoku is a mind bender like no other. It is said to have become the most popular newspaper amusement so far and surprisingly, the great majority of addicts are not the mathematically inclined!



The number 'X' should not be repeated in the lines 'A-A' or 'B-B' or anywhere in the shaded sub-square



The puzzle setter or the player puts numbers in some places - which blocks those rows/columns/sub-squares for those numbers

Mathematicians step in

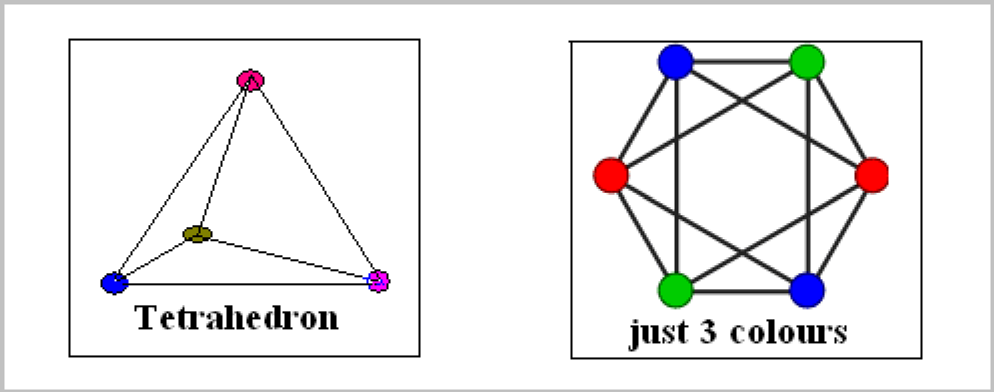
But mathematicians could not be kept away and Agnes M. Herzberg and M. Ram Murty, professors at Queen’s University, Canada, in a paper they have just published in the Journal of the American Mathematical Society, have addressed the questions that perceptive Sudoku junkies have been losing sleep over!

Even before this Canada pair, mathematicians had seen the Sudoku was a kind of ***Latin square***, where each number occurs only once in its row or column. It was worked out that with a 9x9 grid, there could be 5,524,751,496,156,892,842,531,225,600 Latin squares, or over 5 billion billion billion. Of course, the Sudoku square puts in another condition – so all Latin squares are not Sudoku squares. But there are still 6,670,903,752,021,072,936,960 or over 6000 million million million real Sudoku squares, as Bertram Felgenhauer and Frazer Jarvis proved in 2005. This is just about a millionth of the Latin squares there are but the difficulty of solving the puzzles remains.

The few numbers that the puzzle setter puts in actually limit the number of possibilities and simplify the problem. But still, puzzle solvers sometimes wonder if there is a mistake and there may be no solution at all. And then, mathematicians wondered, as every number that is provided reduces the number of solutions, how many such givens will lead to only one possible solution?

Chromatic graphs

A graph, in modern mathematics, is a collection of points and the lines that connect them. In a tetrahedron, like in the picture, there are four vertices and at each vertex three edges meet. Now if we make a rule that vertices of the same colour cannot be connected, then it is clear that we need four different colours in the tetrahedron, because each vertex is connected to the others. But we can see that in the more complex figure beside the tetra, we need only three colours. Satisfying the rule is called proper coloring and it is of interest to discover the least number of colours we need.



Herzberg and Murty considered the Sudoku network as a kind of graph – each number was connected to eight others along the row, with eight others along the column and eight others within the sub-square – and none of these connected numbers could be the same number – or no adjacent vertex could be the same colour!

Making this analogy brought the collection of knowledge about graphs to the disposal of studying Sudodu – and many problems were settled by mathematical methods. By graph theory, for instance, it is known that for a graph of so many vertices to be coloured by so many colours, the number of ways it can be done depends in a certain way on the numbers of vertices and colours. Now, Herzberg and Murty proved that given a partial colouring, or only some vertices coloured to satisfy the condition, then, the number of ways that the job could be completed, using some greater number of colours, also depends in the same way on the number of colours.

This directly touches the question of the number of solutions that are possible when some places in the Sudoku square are marked, by considering each number to represent a colour. Herzberg and Murty point out, for instance, that the least number of places that need to be marked in a Sudoku diagram, so that only one solution exists cannot be more than 17. It is not clear if just 16 places can do the job, but it is known that with 17 places marked, there can still be over 36,000 different solutions.

The Sudoku problems set in newspapers generally have over 24 places marked, to make it simpler for the reader – there are less places to try out and less impossible lines of attack to eliminate. But the Sudoku problems, which are nothing but graph colouring problems, represent different kinds of real-life scheduling problems.

In scheduling meetings, for instance, pairs of committees are connected by an edge (adjacent) if they have a common member. The meetings then cannot be at the same time (colour). But unconnected committees can meet at the same time. Or for instance, assigning frequencies to TV cahnnels. If they are within some distance of each other, they need different frequencies. If some operators already have assigned channels, then accommodating new operators amounts to completing the colouring where a partial colouring has been done, or of completing the Sudoku square where some places are assigned.